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E-mail: ijsi@iscas.ac.cn
http://www.ijsi.org
Tel: +86-10-62661040

Quantum Pushdown Automata with Classical Stack and Tape Head

Jismy Joseph[1] and K. S. Dersanambika[2]

[1] (Mar Thoma College, Tiruvalla, Kerala, India)
[2] (Fatima Mata National College, Kollam, Kerala, India)

Abstract The quantum computational models were proposed to study how quantum mechanics influence the power of computing models. A lot of works are already done to extend simpler classical computational models to quantum models like quantum finite automata[3] and quantum pushdown automata[2,5]. Since the quantum part of a machine is not easy to implement, it is necessary to think about a computational model which minimize the quantum part of the model. Motivated by this fact in this paper we introduce a variation of quantum pushdown automata whose stack and tape head are implemented as classical devices. We observe that this model is powerful than classical pushdown automata and some other quantum computational models. Here we also showed that it can recognize some non context free languages.
Key words: quantum computational models; 1QCFA; QCPA; QCPACT

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1 Introduction

Since Richard Feynman conjectured that no classical computer could efficiently simulate a quantum system, the research on quantum computing has developed and has achieved some important breakthroughs like al2rithms proposed by Shor and Grover, which can solve some important problems more efficiently than their known classical counterparts. But a quantum computer that would be able to implement these theoretical model is not feasible yet and is still a challenge for Physicist and Engineers. So in order to study the power of quantum mechanics in computations, other simpler classical computational models have also been extended to quantum models such as quantum finite automata[1,3,4] and quantum pushdown automata[2,4].

pushdown automata is a very important finite model of computation in the theory of automata. The first definition of quantum model of pushdown automata, quantum pushdown automata was suggested by Moore and Crutchfield in Ref. [4]. But that was a generalized model of quantum pushdown automata, in which the evolution does not have to be unitary. But the basic postulates of quantum mechanics imposes a strong constraint on any quantum machine model that it has to be unitary. So the notion of quantum pushdown automata was reintroduced in Ref. [2] by giving a definition

Corresponding author: Jismy Joseph, Email: jismykjoseph@gmail.com
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which confirmed the unitary requirement. In that paper it was proved that quantum pushdown automata can recognize every regular languages and also it can recognize some non context free languages. But that model contains a quantum tape head and a quantum stack. So it requires $\mathbf{O}(n)$ qubits to store the information.

Since the quantum resources are not cheap and operations are not easy to implement it is necessary to minimize the size of the quantum part. One such "hybrid" variation of pushdown automata was introduced in Ref. [5]. In that paper they introduced and studied about quantum pushdown automata with classical stack. They proved that quantum pushdown automata with classical stack can recognize every deterministic context free languages and also it can recognize some non context free languages. But that model also could implemented with $\lceil logn \rceil$ qubits of information, where n is the input length. That is the size of the quantum part of this computational model also depends on the length of the input.

It would be nicer to think about a quantum model in which the size of quantum part does not depends on the length of the input. Motivated by this idea in this paper we introduce quantum pushdown automata with classical stack and tape head. Informally it can be described as a pushdown automata that has access to a fixed size quantum register upon which it can perform quantum transformations and measurements. In this paper we proved that this model can recognize every context free languages and languages recognized by one way quantum finite automata with classical states[6]. We also showed that it can simulate quantum pushdown automata with classical stack. At the end of this paper we give an example of a non context free language recognized by this model.

2 Preliminaries

In this section we recall the definitions of pushdown automata, $1QCFA$ and $QCPA$.

Definition 2.1. A deterministic pushdown automata is a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$$

where,

- Q is a finite set of states;
- Σ is a finite set of input alphabet;
- Γ is an alphabet, called stack alphabet;
- $q_0 \in Q$ is the initial state;
- $Z \in \Gamma$ is the stack bottom symbol;
- F is the set of final states;
- δ is a mapping from $Q \times \Sigma \times \Gamma$ to finite subsets of $Q \times \Gamma^*$.

The configuration of a PDA at a given instant is denoted by (q, w, γ), where q is a state, w is the string of input symbol and γ is the string of stack symbols.

For a PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$ the language accepted by the PDA is defined as
$L(M) = \{w|(q_0, w, Z) \vdash^* (p, \epsilon, \gamma)\ for\ some\ p \in F\ and\ \gamma \in \Gamma^*\}$.
A PDA recognize the set of deterministic context free languages.

Definition 2.2. A one-way quantum finite automata with quantum and classical states (1QCFA) is a 10-tuple

$$M = (Q, S, \Sigma, \Theta, \Delta, \delta, |q_0>, s_0, S_{acc}, S_{rej})$$

where,

- Q is a set of quantum states;
- S is the set of classical states;
- Σ finite set of alphabet including the left end marker # and right end marker $;
- $|q_0> \in Q$ is the initial quantum state;
- $s_0 \in S$ is the initial classical state; $S_{acc} \subset S$ and $S_{rej} \subset S$ the sets of classical accepting and rejecting states respectively with $S_{acc} \cap S_{rej} = \emptyset$;
- $\Theta(s, \gamma)$ is the unitary transformation on the current quantum state for each $(s, \gamma) \in S \times \Sigma$;
- $\Delta(s, \gamma)$ corresponds to measurement of the quantum state;
- δ is the transition function of the classical states. $\delta(s, \gamma)$ is a mapping from the set of possible results of measurement to S.

Let $L \subset \Sigma^*$ and $0 \leqslant \epsilon < 1/2$ then $1QCFA\ M$ recognize L with one sided error ϵ if

- For any $w \in L$, $P[Macceptsw] = 1$ and
- For any $w \notin L$, $P[Mrejectsw] \geqslant 1 - \epsilon$.

Definition 2.3. A Quantum Pushdown Automata with classical stack (QCPA) is a 11-tuple

$$M = (Q, S, \Sigma, \Gamma, \Theta, \delta, q_0, s_0, \mathcal{O}, S_{acc}, S_{rej})$$

where,

Q and S are the finite sets of quantum and classical states;

Σ and Γ are as defined in the above definitions;

q_0 and s_0 are the initial quantum state and classical state respectively.

$\mathcal{O} = \oplus_i E_i$ is the observable;

$S_{acc} \subset S$ and $S_{rej} \subset S$ are the sets of accepting states and rejecting states respectively such that $S_{acc} \cap S_{rej} = \mathbb{O}$;

Θ is the quantum state transition function from $Q \times \Sigma \times \Gamma$ to $Q \times \{0,1\}$ and $\delta(s,\gamma)$, where $(s,\gamma) \in S \times \Gamma$ is the classical state transition function from set of all possible results of measurement to $S \times \Gamma^*$.

Language recognition of QCPA is same as that of $1QCFA$.

3 Definitions

Informally, we can describe a QCPACT as classical pushdown automata that has access to a quantum register, upon which it can perform quantum transformations and measurements. The transformations and measurements are determined by a local description of the classical portion of the machine, and the result of measurements can determine the manner in which the classical part of the machine evolves. Now we give a formal definition of a QCPACT as follows.

Definition 3.1. A quantum pushdown automata with classical stack and tape head is defined as a 11-tuple

$$M = (Q, S, \Sigma, \Gamma, \Theta, \delta, q_0, s_0, \mathcal{O}, S_{acc}, S_{rej})$$

where,

- Q is the set of quantum states.
- S is the set of classical states.
- Σ is the finite set of alphabet including left end marker # and right end marker $;
- Γ is the set of stack symbols including the bottom symbol Z;
- Θ is the evolution operator for the quantum states;
- δ is the transition function for the classical states;
- q_0 is the initial quantum state;
- s_0 is the initial classical state;
- $S_{acc} \subset S$ and $S_{rej} \subset S$ are the sets of accepting tates and rejecting states respectively such that $S_{acc} \cap S_{rej} = \mathbb{O}$;
- $\mathcal{O} = \oplus_i E_i$ is the observable.

The function Θ specifies the evolution of the quantum states of the system. For each $(s,\alpha,\beta) \in S \setminus (S_{acc} \cup S_{rej}) \times \Sigma \times \Gamma$, $\Theta(s,\alpha,\beta)$ is the unitary transformation performed on the current quantum state of the system. The transition function for the classical part of the system $\delta(s,\alpha,\beta)$ is a mapping from the set of all possible results of the measurement to $S \times \Gamma^* \times \{0,1\}$. It is assumed that δ is defined so that the tape head never moves right when it scans the right end marker.

On a given input x a $QCPACT$ M is to operate as follows. Initially the classical state of M is in s_0 and quantum state of M is in q_0. The tape head of M is scanning the square indexed by 0. The tape squares indexed by $1, 2, \cdots, |x| = n$ contains $x_1, x_2, \cdots, x_n$ while the square indexed by 0 and $n+1$ contains end markers # and \$ respectively. On each step the quantum state changed according to $\Theta(s, \alpha, \beta)$. Then we measure the quantum state of the machine with respect to the observable $\mathcal{O}$. According to the possible result of the measurement the classical state, stack symbol and tape head position changed using $\delta(s, \alpha, \beta)$. If we do not need a measurement assume that we apply identity operator on the quantum states and result of measurement is ϵ with certainty. Since the results obtained from each measurements are probabilistic the transition among the classical part of a given $QCPACT$ may be probabilistic as well.

A computation is assumed to halt if and only if an accepting or rejecting classical state is entered. Let $L \subset \Sigma^*$ and $0 \leqslant \epsilon < 1/2$. A QCPACT M recognize L with one sided error if

1. $P[M accepting w] = 1$ for every $w \in L$ and
2. $P[M accepting w] \geqslant 1 - \epsilon$ for every $w \notin L$.

Other notions of language recognition such as two sided error, zero error etc. may be defined analo2usly, but in this paper we will consider only one sided error.

4 On the Power of QCPACT

In this section we show that QCPACT can recognize languages that are recognizable by PDA, 1QCFA and QCPA. Also we show that QCPACT can recognize the non-context free language $\{a^n b^n c^n | \, n \geqslant 0\}$.

Theorem 4.1. QCPACT can recognize every context free languages.

Proof: We know that for every context free language there exist a PDA recognizing the language. So we prove this theorem by constructing a QCPACT which will simulate the PDA which recognize the given context free language. Let $M_1 = (S_1, \Sigma_1, \Gamma_1, \delta_1, s_1, Z, F)$ be the given PDA. Then we construct the $QCPACT$ as follows.

$$M = (Q, S, \Sigma, \Gamma, \Theta, \delta, q_0, s_0, \mathcal{O}, S_{acc}, S_{rej})$$

$Q = \{q_0\}$, $S = S_1 \cup \{s_{rej}\}$, $\Sigma = \Sigma_1$, $\Gamma = \Gamma_1$,

$\mathcal{O} = E$, where $E = span\{|q_0 >\}$ $S_{acc} = F$, $S_{rej} = \{s_{rej}\}$

The evolution operator is defined as $\Theta(s, \alpha, \beta) = I$, where I is the identity operator, for every $(s, \alpha, \beta) \in S \setminus (S_{acc} \cup S_{rej}) \times \Sigma \times \Gamma$. Classical part transition function δ is defined as

$$\delta(s, \alpha, \beta)(q) = \begin{cases} (\delta_1(s, \alpha, \beta), Z, 1), \; if \; \alpha \neq \$ \\ (\delta_1(s, \alpha, \beta), Z, 0), \; if \; \alpha = \$ \; and \; \delta_1(s, \alpha, \beta) \in F \\ (s_{rej}, \gamma, 0), \; if \; \alpha = \$ \; and \; \delta_1(s, \alpha, \beta) \notin F \end{cases}$$

where q is result of measurement.

In this construction of the $QCPACT$ the quantum state remains the same till the end of the process since the evolution operator we applied on the quantum state is identity operator. The classical part transition function is working according to the transition function of the pushdown automata that recognize the given context free language. The tape head is moving right till the end marker is reached or the classical state of the machine enters the final state. Thus the constructed $QCPACT$ will recognize the given context free language. □

Theorem 4.2. QCPACT can recognize every language recognizable by 1QCFA

Proof: Let $M_1 = (Q_1, S_1, \Sigma_1, \Theta_1, \mathcal{O}_1, \delta_1, q_0, s_0, S_{acc}, S_{rej})$ be a 1QCFA. Then we can construct a $QCPACT$ as follows which can simulate the given $1QCFA$.

$$M = (Q, S, \Sigma, \Gamma, \Theta, \delta, q_0, s_0, \mathcal{O}, S_{acc}, S_{rej})$$

$Q = Q_1$, $S = S_1$, $\Sigma = \Sigma_1$, $\Gamma = \{Z\}$

Unitary transformation and classical part transition function is defined as follows. $\Theta(s, \alpha, \beta) = \Theta_1(s, \alpha)$, *for every* $(s, \alpha, \beta) \in S \setminus (S_{acc} \cup S_{rej}) \ \times \Sigma \times \Gamma$

$$\delta(s, \alpha, \beta)(q) = \begin{cases} (\delta_1(s, \alpha), Z, 1), \alpha \neq \$ \\ (\delta_1(s, \alpha), Z, 0), \alpha = \$ \end{cases}$$

The quantum states of the $QCPACT$ M will change according to the evolutionary operator of the $1QCFA$. The classical part will change according to the classical part transition function of M_1. There is no change will occur on the stack symbol during the transition. Thus the $QCPACT$ M will clearly recognize the language recognized by M_1. □

Theorem 4.3. A $QCPA$ can be simulated by a $QCPACT$.

Proof: Let

$$M_1 = (Q_1, S_1, \Sigma_1, \Gamma_1, \delta_1, \Theta_1, q_0, s_0, \mathcal{O}_1, S_{acc}, S_{rej})$$

be the $QCPA$. We can construct the $QCPACT$ which will simulate the given $QCPA$ as follows.

$$M = (Q, S, \Sigma, \Gamma, \Theta, \delta, q_0, s_0, \mathcal{O}, S_{acc}, S_{rej})$$

$Q = Q_1$, $\Sigma = \Sigma_1$, $S = S_1$, $\Gamma = \Gamma_1$. The unitary transformation and classical part transition function can be defined as follows.

$$\Theta(s, \alpha, \beta) = \pi_1(\Theta_1(q, \alpha, \beta))$$

where q is the possible result of measurement.

$$\delta(s, \alpha, \beta)(q) = (\delta_1(s, \alpha, \beta), \pi_2(q, \alpha, \beta))$$

where q is the possible result of measurement. π_1 and π_2 are the projection map to the first and second component respectively. □

The following example shows that $QCPACT$ can also recognize some non-context free language.

Example 1. For any positive integer $N > 0$ there exist a $QCPACT$ that accepts any $x \in \{a^n b^n c^n |\ n \geqslant 0\}$ with certainty and rejects
$x \notin \{a^n b^n c^n |\ n \geqslant 0\}$ with probability at least $1 - 1/N$.
We can construct the $QCPACT$ as follows For any positive integer N define M_N as

$$M_N = (Q, S, \Sigma, \Gamma, \Theta, \delta, q_0, s_0, \mathcal{O}, S_{acc}, S_{rej})$$

where,

$$
\begin{aligned}
Q &= \{q_0, q_1\} \\
&\quad \cup \{r_{j,k} | \; 1 \leqslant j \leqslant N, \; 0 \leqslant k \leqslant j\} \\
&\quad \cup \{r'_{j,k} | \; 1 \leqslant j \leqslant N, \; 0 \leqslant k \leqslant N - j + 1\} \\
&\quad \cup \{s_{j,k} | \; 1 \leqslant j \leqslant N, \; 0 \leqslant k \leqslant j\} \\
&\quad \cup \{s'_{j,k} | \; 1 \leqslant j \leqslant N, \; 0 \leqslant k \leqslant N - j + 1\} \\
&\quad \cup \{s''_j | \; 1 \leqslant j \leqslant N\} \\
\Sigma &= \{a, b, c\}, \;\; \Gamma = \{A, B, Z\}, \;\; Q_{acc} = \{S''_N\} \\
Q_{rej} &= \{q_1\} \\
&\quad \cup \{s_{j,k} | \; 1 \leqslant j \leqslant N, \; 0 \leqslant k \leqslant j\} \\
&\quad \cup \{s'_{j,k} | \; 1 \leqslant j \leqslant N, \; 0 \leqslant k \leqslant N - j + 1\} \\
&\quad \cup \{s''_j | \; 1 \leqslant j < N\} \\
E_{non1} &= span\{|q_0 >\} \\
E_{non2} &= span\{|q > |q \in Qn(Q_{acc} \cup Q_{rej} \cup \{q_0\})\} \\
E_{acc} &= span\{|q > |q \in Q_{acc}\} \\
E_{rej} &= span\{|q > |q \in Q_{rej}\}
\end{aligned}
$$

$\mathcal{O} = E_{non1} \oplus E_{non2} \oplus E_{acc} \oplus E_{rej}$.

The working of M_N is described by the following algorithm.

1. If the current symbol is # move the tape head one square to the right.
2. If the current symbol is b or c reject.
3. If the current symbol is \$ accept.
4. While the current symbol is a push A to the stack and move the tape head one square to the right.
5. If the current symbol is c or \$ reject.
6. If the current symbol is b apply $\frac{1}{\sqrt{N}} \sum_{j=1}^{N} |r_{j,0} >$ to the quantum state.
7. Pop A from the stack and stay in the same tape head position.
8. Change the quantum state $|r_{j,0} >$ to $|r_{j,1 \; mod \; j+1} >$.
9. While the result of measurement is not $|r_{j,0} >$ change the quantum state $|r_{j,k} >$ to $|r_{j,k+1 \; mod \; j+1} >$ and measure the quantum state.
10. Move the tape head one square to the right. If the current symbol is b 2to 7.
11. If the current symbol is c and stack is nonempty reject.
12. If the current symbol is c and stack is empty change the quantum state $|r_{j,0} >$ to $|r'_{j,0} >$ and stay in the same tape head position.
13. Change the quantum state to $|r'_{j,1 \; mod \; N-j+1} >$.

14. While the result of measurement is not $|r'_{j,0}>$ stay there in the same tape head position and change the quantum state $|r'_{j,k}>$ to $|r'_{j,k+1 \ mod \ N-j+1}>$.

15. Move the tape head one square to the right. If the current symbol is c 2to 13.

16. If the current symbol is a or b then reject.

17. If the current symbol is \$ apply $\frac{1}{\sqrt{N}}\sum_{l=1}^{N}\exp(\frac{2\pi i}{N}jl)|s''_l>$ to the quantum state.

18. If the result of measurement is $|S''_N>$, accept otherwise reject.

5 Conclusion

In this paper we introduce and study about the quantum pushdown automata model $QCPACT$ which is a pushdown automata with a constant quantum register. We proved that $QCPACT$ can simulate the quantum pushdown automata with classical stack, whose size of quantum part does not depends on the input length. And also we give an example of a non context free language $\{a^nb^nc^n \mid n \geqslant 0\}$ recognized by $QCPACT$.

It is still a future work to study whether $QCPACT$ is equivalent to $QCPA$. And also we want to check whether the power of $QCPACT$ can increase by adding a two way tape head.

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